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Moment Inequalities and High-Energy Tails for Boltzmann Equations with Inelastic Interactions

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We study high-energy asymptotics of the steady velocity distributions for model kinetic equations describing various regimes in dilute granular flows. The main results obtained are integral estimates of solutions of the Boltzmann equation for inelastic hard spheres, which imply that steady velocity distributions behave in a certain sense as $C\exp(-r|v|^s)$, for $|v|$ large. The values of s, which we call *the orders of tails*, range from $s = 1$ to $s = 2$, depending on the model of external forcing. To obtain these results we establish precise estimates for exponential moments of solutions, using a sharpened version of the Povzner-type inequalities.

KEY WORDS: Boltzmann equations; inelastic collisions; granular flows; heat bath; shear flow; high-energy tails; Povzner inequalities.

1. INTRODUCTION

In this paper we address the problem of high-energy asymptotics for solutions of kinetic equations used for modeling dilute, rapid flows of granular media. Granular systems in such regimes are interesting from a physical point of view, since they show a variety of interesting and unexpected properties. They also appear in a growing number of industrial applications. Much of the interest to kinetic models in this context comes from the fact that such models provide a systematic way of derivation of hydrodynamic equations based on the principles of particle dynamics.

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They are also useful for numerical modeling of granular flows. We refer the reader to the review papers[10,21,22] for a general exposition of the subject.

In dilute flows, binary collisions are often assumed to be the main mechanism of the particle interactions. The effect of such collisions can be modeled by collision terms of the Boltzmann or Enskog type. An important feature of the particle interactions is their inelastic character: a certain fraction of the kinetic energy of the particles is dissipated in every collision. This introduces some interesting features in the equations: in particular, the only functions on which the collision operator vanishes are delta functions corresponding to all particles at rest (or moving with the same velocity).

To obtain other nontrivial steady states in granular systems, as a general rule, a certain mechanism of the energy inflow is required. Experimentally this can be achieved, for example, by shaking a vessel with granular particles. In terms of equations, several simplified models have been proposed, in the spatially homogeneous scenario, which include forcing terms of various types.[36,33,15] Examples of such terms are diffusion (in the velocity space) and Fokker-Planck operators which correspond to the model of granular particles in a thermal bath.

Other important types of problems which lead to similar equations are related to self-similar solutions in the homogeneous cooling problem and the problem of shear flow.[17,11,9] In both cases the equations can be transformed, after an appropriate change of variables, to spatially homogeneous steady problems for the Boltzmann-type equations with force terms that correspond to the negative or anisotropic friction.

One of the interesting features of granular flows, which can be studied in the framework of the mentioned models, is the non-Maxwellian behavior of the steady velocity distributions. In fact, experimental data, theoretical predictions and numerical evidence suggest that typical velocity distributions in rapid granular flows have high-energy asymptotics (or "tails") given by the "stretched exponentials" $\exp(-r|v|^s)$ with s generally not equal to 2 (the classical, Maxwellian, case), or display power-like decay for $|v|$ large (see refs. 19 and 15 and references therein).

The precise form of the asymptotics is determined by several factors, among which are the details of the interactions and the forcing models. In the present paper we study the model spatially homogeneous system with collisions of the *hard-sphere* type and four variants of forcing terms. We consider the cases of (i) diffusion (Gaussian heat bath), (ii) diffusion with friction (Fokker-Planck type terms), (iii) negative friction (obtained in a self-similar transformation in the homogeneous cooling problem, and (iv) anisotropic friction which appears in the shear flow transformation.

By studying the *exponential moments* of solutions (functionals of the form Eq. (2.17)) we obtain information about high-energy tails of the steady solutions of the models. The form of the tails is given by "stretched exponentials" $\exp(-r|v|^s)$, with s depending on the forcing terms. We obtain the values $s=3/2$ for the pure diffusion case, $s=2$ for the diffusion-friction heat bath, $s=1$ for the negative friction case and $s\geqslant 1$ in the case of the shear flow. Our method is based on studying the families of moments of the distribution functions and establishing precise bounds for these families using a variant of the so-called Povzner inequalities, similar to the one studied by one of the authors(4) in the case of the classical (elastic) spatially homogeneous Boltzmann equation. We expect that the estimates obtained for the moments can be used to describe the behavior of the tails in the time-dependent case, which should be an object of a separate study.

The problem of high-energy tails for the hard-sphere Boltzmann models has been studied previously by several authors(17,33,16,15) by the methods of formal asymptotic analysis. A formal argument becomes particularly simple if one discards the "gain" term in the equations; this is motivated by the observation(17,33) that for $h(v)=C\exp(-r|v|^s)$, with $s<2$, the "gain" term $Q^+(h,h)$ in the collision operator is a small perturbation of the loss term for $|v|$ large. The moment method developed in the present work gives a new and completely rigorous interpretation of this principle, based on the properties of solutions of the original problem. The problem with diffusive forcing has been analyzed in ref. 19, where it was proved, in particular, that steady solutions are infinitely differentiable and decay faster than any polynomial for $|v|$ large. A lower bound for the steady solutions by $C\exp(-r|v|^{3/2})$ was also established by using a comparison principle. The problem has also been studied numerically by a number of authors.(29,3,30,20)

Another series of related results was obtained for the so-called inelastic Maxwell models,(5,24,26) which are approximate equations obtained by replacing the collision kernel in the Boltzmann operator by a certain mean value independent on the relative velocity. This particular Fourier transform structure of the collision kernel allows one to take advantage of the powerful Fourier transform methods; moreover, the equations for symmetric moments form a closed infinite recursive system, similarly to the Boltzmann equation for elastic particles.(5,26) Using the Fourier transform techniques, Bobylev and Cercignani(6) found solutions to the inelastic Maxwell model with a heat bath, which have high-energy tails $\exp(-r|v|)$. For the self-similar scaling problem, solutions with power-like tails were found,(2,14,25) and it was conjectured by Ernst and Brito(16) that such solutions determine the universal long-time asymptotics of the time-dependent solutions in the spatially homogeneous cooling problem. This conjecture has recently been proved by Bobylev, Cercignani and Toscani.[7,8]

It should be noted, however, that while Maxwell models may give reasonable approximations of the macroscopic quantities, the details of the velocity distributions can differ significantly from the hard-sphere case. In particular, this is true with respect of the high-energy asymptotics which depends crucially on the behavior of collision rate for large relative velocities, as can be easily seen from the formal asymptotic arguments of the type presented in refs. 15, 17, and 33. Therefore, the aim of this paper is to develop rigorous methods that would allow us to study solutions of the hard-sphere Boltzmann equation for inelastic particles, with a particular attention to the high-energy asymptotics.

The paper is organized as follows. In Section 2 we formulate the problem and state the main results. In Section 3 we develop an approach to Povzner-type inequalities which applies to both elastic and inelastic collisions and which allows us to obtain precise constants in the moment inequalities. The results of that section are the key to subsequent estimates of the exponential moments. Section 4 presents the inequalities for the symmetric moments in the form that is specific to the hard-sphere model. In Section 5 we find the inequalities for the normalized moments which appear as the coefficients of power series expansions of functionals (2.17). We further study the dependence of the inequalities on the parameters and find conditions under which the sequences of the normalized moments have geometric growth. Finally, Section 6 presents the proofs of the main theorems.

Most of our inequalities can be used in the time-dependent case, and therefore, we begin the next Section by considering the non-stationary Boltzmann equation.

2. PRELIMINARIES AND MAIN RESULTS

We study kinetic models for spatially homogeneous granular media, in which the one-particle distribution function $f(v,t)$, $v \in \mathbb{R}^3$, $t \geqslant 0$ is assumed to satisfy the following equation:

$$\frac{\partial f}{\partial t} = \mathcal{Q}(f, f) + \mathcal{G}(f). \tag{2.1}$$

Here $\mathcal{Q}(f, f)$ is the inelastic Boltzmann collision operator, expressing the effect of binary collisions of particles, and $\mathcal{G}(f)$ is a forcing term. We will consider three different examples of forcing. The first one is the pure diffusion thermal bath,[36,33,19] in which case

$$\mathcal{G}_1(f) = \mu\,\Delta f, \tag{2.2}$$

where $\mu > 0$ is a constant. The second example is the thermal bath with linear friction

$$\mathcal{G}_2(f) = \mu\,\Delta f + \lambda \operatorname{div}(v\,f), \tag{2.3}$$

where λ and μ are positive constants.

The third example relates to self-similar solutions of Eq. (2.1) for $\mathcal{G}(f) = 0$.[29,15] We denote

$$f(v,t) = \frac{1}{v_0^3(t)}\,\tilde{f}\big(\tilde{v}(v,t), \tilde{t}(t)\big), \quad \tilde{v} = \frac{v}{v_0(t)},$$

where

$$v_0(t) = (a + \kappa t)^{-1}, \quad \tilde{t}(t) = \frac{1}{\kappa}\ln(1 + \frac{\kappa}{a}t), \quad a,\, \kappa > 0.$$

Then, the equation for $\tilde{f}(\tilde{v}, \tilde{t})$ coincides (after omitting the tildes) with Eq. (2.1), where

$$\mathcal{G}_3(f) = -\kappa \operatorname{div}(vf), \qquad \kappa > 0. \tag{2.4}$$

Finally, the last type of forcing is given by the term appearing in the shear flow transformation (see, for example, refs. 11 and 9)

$$\mathcal{G}_4(f) = -\kappa\, v_1 \frac{\partial f}{\partial v_2}, \tag{2.5}$$

where κ is a positive constant.

To define the collision operator we set

$$\mathcal{Q}(f,f) = \mathcal{Q}^+(f,f) - \mathcal{Q}^-(f,f), \tag{2.6}$$

where $\mathcal{Q}^-(f,f)$ is the "loss" term:

$$\mathcal{Q}^-(f,f) = \int_{\mathbb{R}^3}\int_{S^2} f(w)\, B(v-w,\sigma)\, d\sigma\, dw, \tag{2.7}$$

and $\mathcal{Q}^+(f,f)$ is the "gain" term which is most easily defined through its weak form: for every suitable test function ψ,

$$\int_{\mathbb{R}^3} \mathcal{Q}^+(f,f)\,\psi\,dv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)\,f(w) \int_{S^2} \psi(v')\,B(v-w,\sigma)\,d\sigma\,dw\,dv. \tag{2.8}$$

Here v' is the velocity assumed by a particle in the collision defined by the velocities v, w and the angular parameter $\sigma\in S^2$ (cf. e.g., ref. 19):

$$\begin{aligned} v' &= v + \frac{\beta}{2}\,(|u|\sigma - u), \\ w' &= w - \frac{\beta}{2}\,(|u|\sigma - u), \end{aligned} \tag{2.9}$$

where $u = v - w$ is the relative velocity, and we set $\beta = \frac{1+e}{2}$, where $0 < e < 1$ is the normal restitution coefficient. Notice that we always have $1/2 < \beta < 1$.

The function $B(v-w,\sigma)$ appearing in Eqs. (2.7) and (2.8) is the collision kernel, which for the hard sphere model can be taken as simply

$$B(u,\sigma) = \frac{1}{4\pi}\,|u|. \tag{2.10}$$

The equation with the kernel (2.10) will be the main subject of this work; however our approach also applies to more general kernels,

$$B(u,\sigma) = |u|^{\zeta}\,b(\cos\vartheta), \quad \cos\vartheta = \frac{(u\cdot\sigma)}{|u|}, \tag{2.11}$$

where $\zeta > 0$, and

$$b(z) \geqslant 0 \text{ is nondecreasing, convex on } (-1,1) \tag{2.12}$$

and satisfies the normalization condition

$$\int_{-1}^{1} b(z)\,dz = \frac{1}{2\pi}. \tag{2.13}$$

Such choice of B is motivated by the model kernels of the "hard forces" type with angular cutoff in the elastic case.

Combining Eqs. (2.7) and (2.8) and using the symmetry that allows us to exchange v with w in the integrals we obtain the following symmetrized weak form

$$\int_{\mathbb{R}^3} Q(f, f)\,\psi\, dv = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)\, f(w)\, A_\beta[\psi](v, w)\, dw\, dv, \tag{2.14}$$

where

$$A_\beta[\psi](v, w) = \int_{S^2} (\psi(v') + \psi(w') - \psi(v) - \psi(w))\, B(u, \sigma)\, d\sigma. \tag{2.15}$$

We will assume that the solutions are normalized as follows

$$\int_{\mathbb{R}^3} f(v, t)\, dv = 1, \quad \int_{\mathbb{R}^3} f(v, t)\, v_i\, dv = 0, \quad i = 1, 2, 3. \tag{2.16}$$

Since the expected behavior of solutions for $|v|$ large is $C\exp(-r|v|^s)$, we introduce the following functionals:

$$\mathcal{F}_{r,s}(f) = \int_{\mathbb{R}^3} f(v)\, \exp(r|v|^s)\, dv, \tag{2.17}$$

and study the values of s and r for which these functionals are finite. This motivates the following definition.

Definition 1. We say that the function f has an *exponential tail of order* $s > 0$ if

$$r_s^* = \sup\left\{r > 0 \mid \mathcal{F}_{r,s}(f) < +\infty\right\} \tag{2.18}$$

is positive and finite.

In the case $s = 2$ the value of $(r_s^*)^{-1}$ is known as the *tail temperature* of f.[4] It is easy to see that the value of s in the above definition is uniquely determined. Indeed, if for certain $s > 0$,

$$0 < r_s^* < +\infty,$$

then $r_{s'}^* = +\infty$, for every $s' < s$, and $r_{s'}^* = 0$, for every $s' > s$.

Another useful representation of the functionals 2.17 is obtained by using the *symmetric moments* of the distribution function. Setting

$$m_p = \int_{\mathbb{R}^3} f(v)|v|^{2p}\, dv, \quad p \geqslant 0, \tag{2.19}$$

and expanding the exponential function in Eq. (2.17) into a Taylor series we obtain (formally)

$$\mathcal{F}_{r,s}(f) = \int_{\mathbb{R}^3} f(v) \Big(\sum_{k=0}^{\infty} \frac{r^k}{k!} |v|^{sk} \Big) dv = \sum_{k=0}^{\infty} \frac{m_{\frac{sk}{2}}}{k!} r^k. \tag{2.20}$$

Then the value r_s^* from Eq. (2.18) can be interpreted as the *radius of convergence* of the series Eq. (2.20), and the order of the tail s is therefore the *unique* value for which the series has a positive and finite radius of convergence.

We can now formulate the main results of this study. Our first result concerns steady states of equation (2.1) corresponding to the first three types of forcing.

Theorem 1. Let $f_i(v)$, $i = 1, 2, 3$, be nonnegative steady solutions of the equations (2.1), with the hard-sphere kernel (2.10), and with the forcing terms (2.2, 2.3 and 2.4), respectively. Assume that $f_i(v)$ have finite moments of all orders. Then $f_i(v)$ have exponential tails of orders $\frac{3}{2}$, 2 and 1, respectively.

For the shear flow model (2.5), we obtain the following weaker result.

Theorem 2. Let $f_4(v)$ be a nonnegative steady solution of the shear flow model (2.1), (2.5), with the hard-sphere kernel (2.10), that has finite moments of all orders. Then the supremum r_1^*, defined in (2.18), is finite, and therefore, $s \geqslant 1$.

Remark 1. The assumption of finiteness of moments of all orders is obviously required for the functionals (2.17) to be finite. However, the moment inequalities we establish below also imply the following *a priori estimates* for all cases of the solutions: *Suppose that a moment m_{p_0} of any order $p_0 > 1$ is finite. Then, in fact, all moments are finite and the solutions have exponential tails of the corresponding order.* This observation is important, since it excludes the possibility of power-like decay for solutions of the considered equations, as soon as solutions have finite mass and finite moment of any order higher than kinetic energy.

The approach that we take in order to establish the above results is based on the moment method, in the form developed by one of the authors,[4] for the classical spatially homogeneous Boltzmann equation. We study the moment equations obtained by integrating (2.1) against $|v|^{2p}$:

$$\frac{dm_p}{dt} = Q_p + G_p \tag{2.21}$$

(in the steady case the time-derivative term drops out), where

$$Q_p = \int_{\mathbb{R}^3} \mathcal{Q}(f, f)\, |v|^{2p}\, dv \quad \text{and} \quad G_p = \int_{\mathbb{R}^3} \mathcal{G}(f)\, |v|^{2p}\, dv. \tag{2.22}$$

To investigate the summability of the series (2.20) we look for estimates of the sequence of moments (m_p), with $p = \frac{sk}{2}$, $k = 0, 1, 2\ldots$ We will be interested in the situation when the series has a finite and positive radius of convergence, which means that the sequence of the coefficients satisfies

$$c\, q^k \leqslant \frac{m_{\frac{sk}{2}}}{k!} \leqslant C\, Q^k, \quad k = 0, 1, 2...,$$

for certain constants $q > 0$ and $Q > 0$.

3. POVZNER-TYPE INEQUALITIES FOR INELASTIC COLLISIONS

In this section we establish the main technical result of the paper, which will be the key for obtaining precise estimates of the moments of the collision terms. We will consider test functions $\psi(v) = \Psi(|v|^2)$, where $\Psi(z)$ is nondecreasing and convex for $z \in (0, \infty)$. Then, for collision kernels of the type (2.11), expression (2.15) can be written as

$$A_\beta[\psi] = |u|^\zeta \left(A_\beta^+[\Psi] - A_\beta^-[\Psi]\right),$$

where the "gain" part $A_\beta^+[\Psi]$ is

$$A_\beta^+[\Psi](v, w) = \int_{S^2} \left(\Psi(|v'|^2) + \Psi(|w'|^2)\right) b(\cos\vartheta)\, d\sigma, \tag{3.1}$$

where $\cos\vartheta = (u \cdot \sigma)/|u|$, $u = v - w$, and the "loss" part $A_\beta^-[\Psi]$ is simply

$$A_\beta^-[\Psi](v, w) = \Psi(|v|^2) + \Psi(|w|^2). \tag{3.2}$$

For the rest of this section we assume that $b(z)$ is nonnegative, nondecreasing, convex and satisfies $\int_{-1}^{1} b(z)\,dz = \frac{1}{2\pi}$. The hard-sphere model is a particular case for which $b(z) = \frac{1}{4\pi}$.

A series of results(4,12,13,23,27,28,31,35) for the classical Boltzmann operator with elastic collisions develops the general idea that for convex Ψ the "gain" part (3.1) is in certain sense "of lower order", for $|v|$ and $|w|$ large, than the "loss" part (3.2). Results of this type are generally known as Povzner-type inequalities. An approach for extending such inequalities to inelastic collisions has recently been developed in ref. 19. However, for the purposes of the present study we need more precise estimates of the constants in the inequalities than those of ref. 19. Therefore, our goal here will be to establish a sharper version of the Povzner inequality for inelastic collisions, using the approach of ref. 4. The two main ideas are to pass to the center of mass – relative velocity variables and to use the angular integration in (3.1) to obtain more precise constants in the inequalities.

We start by setting $U = \frac{u+w}{2}$, the velocity of the center of mass, and $u' = v' - w'$, so that

$$v' = U + \frac{u'}{2}, \quad w' = U - \frac{u'}{2},$$

where, according to (2.9),

$$u' = (1-\beta)u + \beta|u|\sigma, \tag{3.3}$$

and $u = v - w$. We further represent $u' = \lambda|u|\omega$, where ω is the unit vector in the direction of u' and λ can be computed according to Eq. (3.3) as

$$\lambda = (1-\beta)(\nu\cdot\omega) + \sqrt{(1-\beta)^2(\nu\cdot\omega)^2 + 2\beta - 1}, \tag{3.4}$$

where $\nu = u/|u|$. This allows us to express σ from Eq. (3.3) as

$$\sigma = \frac{\lambda\omega - (1-\beta)\nu}{\beta}. \tag{3.5}$$

We further pass to the integration $d\omega$ in the integral (3.1), for which we notice that for every suitable test function φ,

$$\begin{aligned}\int_{S^2} \varphi(\sigma)\, d\sigma &= \int_{\mathbb{R}^3} \varphi(k)\, \delta\Big(\frac{|k|^2-1}{2}\Big)\, dk \\ &= \frac{1}{\beta^3}\int_{S^2}\int_0^\infty \rho^2\, \varphi\Big(\frac{\rho\omega-(1-\beta)\nu}{\beta}\Big) \\ &\qquad \times \delta\Big(\frac{|\rho\omega-(1-\beta)\nu|^2-\beta^2}{2\beta^2}\Big)\, d\rho\, d\omega \\ &= \int_{S^2} \varphi\Big(\frac{\lambda\omega-(1-\beta)\nu}{\beta}\Big)\, g_\beta(\nu\cdot\omega)\, d\omega, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} g_\beta(\nu\cdot\omega) &= \frac{1}{\beta^3}\int_0^\infty \rho^2\, \delta\Big(\frac{|\rho\omega-(1-\beta)\nu|^2-\beta^2}{2\beta^2}\Big)\, d\rho \\ &= \frac{1}{\beta}\int_0^\infty \rho^2\, \delta\Big(\frac{(\rho-(1-\beta)\mu)^2-2\beta+1}{2}\Big)\, d\rho \\ &= \frac{\big((1-\beta)\mu+\sqrt{(1-\beta)^2\mu^2+2\beta-1}\big)^2}{\beta\sqrt{(1-\beta)^2\mu^2+2\beta-1}}, \end{aligned}$$

and we denoted $\mu=(\nu\cdot\omega)$. Notice that for every $\beta\in[\frac{1}{2},1]$, the function $g_\beta(\mu)$ for $\mu\in(-1,1)$ is positive, nondecreasing, convex and satisfies the normalization condition $\int_{-1}^{1} g_\beta(\mu)\, d\mu = 2$.

Applying identity (3.6) to the integral in (3.1) we obtain

$$A_\beta^+[\Psi] = \int_{S^2} \Big\{\Psi\Big(\Big|U+\lambda\frac{|u|}{2}\omega\Big|^2\Big)+\Psi\Big(\Big|U-\lambda\frac{|u|}{2}\omega\Big|^2\Big)\Big\}\, g_\beta(\mu)\, b_\beta(\mu)\, d\omega, \tag{3.7}$$

where we denoted

$$b_\beta(\mu) = b\Big(\tfrac{\mu\lambda(\mu)}{\beta} - \tfrac{1-\beta}{\beta}\Big) = b(\cos\vartheta).$$

The function $\mu\lambda(\mu)$ can be shown to be nondecreasing and convex for $\mu\in(-1,1)$, for each $\beta\in[\frac{1}{2},1]$. Since we assumed $b(z)$ to be nondecreasing and convex on $z\in(-1,1)$, the same holds for the function $g_\beta(\mu)\, b_\beta(\mu)$. One can also verify by taking $\Psi=\frac{1}{2}$ in 3.7 that

$$\int_{-1}^{1} g_\beta(\mu)\, b_\beta(\mu)\, d\mu = \frac{1}{2\pi}.$$

We can now formulate the following version of the Povzner inequality.

Lemma 1. Assume that the function $b(z)$ in Eq. (2.11) satisfies Eqs. (2.12) and (2.13). For every $\beta \in [\frac{1}{2}, 1]$, and for every function $\Psi(x)$, $x > 0$, such that $\Psi(x)$ is everywhere finite, nondecreasing and convex,

$$A_\beta^+[\Psi] \leqslant 4\pi \int_{-1}^{1} \Psi\Big((|v|^2 + |w|^2)\, \frac{1+\mu}{2}\Big)\, h_\beta(\mu)\, d\mu\,,$$

where $h_\beta(\mu)$ is the symmetric part of $g_\beta(\mu)\, b_\beta(\mu)$:

$$h_\beta(\mu) = \frac{1}{2}\, \big(g_\beta(\mu)\, b_\beta(\mu) + g_\beta(-\mu)\, b_\beta(-\mu)\big).$$

Remark 2. The above inequality is a generalization of inequalities Eqs. (12) and (16) from ref. 4 to the inelastic case, under the extra assumption of Ψ being nondecreasing. (For the elastic hard spheres, $h_\beta(\mu) = \frac{1}{4\pi}$, and we verify the result of ref. 4)

Proof. The proof relies on the following property of convex functions: suppose that $\psi(x)$, $x \in \mathbb{R}^n$ is convex and finite for every x. Then, for every $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$,

$$\psi(x + ty) + \psi(x - ty) \tag{3.8}$$

is a nondecreasing function of $t > 0$. Indeed, since $\varphi(t) = \psi(x + ty)$ is convex in t, we can write

$$\psi(x + ty) + \psi(x - ty) = \varphi(t) + \varphi(-t) = \int_0^t \big(\varphi'_+(\tau) - \varphi'_+(-\tau)\big)\, d\tau, \tag{3.9}$$

where φ'_+ is the right-sided derivative of φ (see ref. 32, Sec. 24). Moreover, since $\varphi'_+(\tau)$ is nondecreasing, Eq. (3.9) implies the required property of Eq. (3.8).

Noticing that $\Psi(|\cdot|^2)$ is a convex function, and that $\lambda = \lambda(\nu \cdot \omega)$ defined by Eq. (3.4) satisfies $\lambda \leqslant 1$, we can apply the above convexity argument and estimate the expression in braces in Eq. (3.7) from above as

$$\Psi\Big(\Big|U + \frac{|u|}{2}\omega\Big|^2\Big) + \Psi\Big(\Big|U - \frac{|u|}{2}\omega\Big|^2\Big) = \Psi\Big(E\, \frac{1+\xi}{2}\Big) + \Psi\Big(E\, \frac{1-\xi}{2}\Big), \tag{3.10}$$

where

$$E = 2U^2 + \frac{|u|^2}{2} = |v|^2 + |w|^2, \qquad \xi = \frac{2|U||u|}{E}\,(m\cdot\omega),$$

and m is the unit vector in the direction of U. Further, since

$$\frac{2|U||u|}{E} \leqslant 1,$$

we can use the convexity of the function $\Psi(E(\frac{1+\cdot}{2}))$, and estimate Eq. (3.10) by

$$\Psi\Big(E\,\frac{1+(m\cdot\omega)}{2}\Big) + \Psi\Big(E\,\frac{1-(m\cdot\omega)}{2}\Big). \tag{3.11}$$

Since Eq. (3.11) is invariant with respect to the change of variables $\omega \mapsto -\omega$, the weight function $g_\beta(\mu)\,b_\beta(\mu)$ in Eq. (3.7) can be replaced by $h_\beta(\mu)$, and hence,

$$A_\beta^+[\Psi] \leqslant \int_{S^2} \Big\{\Psi\Big(E\,\frac{1+(m\cdot\omega)}{2}\Big) + \Psi\Big(E\,\frac{1-(m\cdot\omega)}{2}\Big)\Big\}\, h_\beta(\nu\cdot\omega)\, d\omega. \tag{3.12}$$

The integral above has the form

$$I(\nu, m) = \int_{S^2} f_1(|\nu\cdot\omega|)\ f_2(|m\cdot\omega|)\ d\omega, \quad \nu, m \in S^2,$$

where both $f_1(z)$ and $f_2(z)$ are nondecreasing on $(0,1)$. The next step is to prove that

$$I(\nu, m) \leqslant I(\nu, \nu) = 4\pi \int_0^1 f_1(z)\ f_2(z)\, dz, \tag{3.13}$$

or equivalently, that

$$\Delta(\nu, m) = \int_{S^2} f_1(|\nu\cdot\omega|)\ (f_2(|\nu\cdot\omega|) - f_2(|m\cdot\omega|))\, d\omega \geqslant 0. \tag{3.14}$$

To prove Eq. (3.14) we notice the symmetry

$$\Delta(\nu, m) = \Delta(m, \nu),$$

since $\Delta(\nu, m)$ is a function of the scalar product $(\nu \cdot m)$ only. Therefore,

$$\begin{aligned}\Delta(\nu, m) &= \frac{1}{2}\left(\Delta(\nu, m) + \Delta(m, \nu)\right) \\ &= \frac{1}{2}\int_{S^2}\big(f_1(|\nu\cdot\omega|) - f_1(|m\cdot\omega|)\big)\big(f_2(|\nu\cdot\omega|) - f_2(|m\cdot\omega|)\big)\,d\omega.\end{aligned}$$

But then Eq. (3.14) follows if we use the inequality

$$\big(f_1(z) - f_1(y)\big)\big(f_2(z) - f_2(y)\big) \geqslant 0, \quad z, y \in (0,1),$$

for monotone functions $f_1(z)$ and $f_2(z)$.

Using the established inequality Eq. (3.13) we obtain

$$\begin{aligned}A^{+}_{\beta}[\Psi] &\leqslant \int_{S^2}\left\{\Psi\left(E\,\tfrac{1+(\nu\cdot\omega)}{2}\right) + \Psi\left(E\,\tfrac{1-(\nu\cdot\omega)}{2}\right)\right\} h_\beta(\nu\cdot\omega)\,d\omega \\ &= 4\pi\int_{-1}^{1}\Psi\left(E\,\tfrac{1+\mu}{2}\right) h_\beta(\mu)\,d\mu\,,\end{aligned}$$

and we arrive at the conclusion of the lemma. ▮

Remark 3. As the careful reader will easily check, the assumption of $\Psi(x)$ being nondecreasing is only used in Eq. (3.10) and is not needed in the elastic case $\beta = 1$ when we have $\lambda \equiv 1$ and $g_\beta(\mu) \equiv 1$. Moreover, the convexity of $b(z)$ can in that case be replaced by a weaker assumption: the symmetric part of $b(z)$ (the sum of the values at z and $-z$), must be nondecreasing on $(0,1)$. Thus, the present result extends the Povzner-type estimate in ref. 4 to collision kernels with monotone angular dependence satisfying the above condition on the symmetric part.

For the functions $\Psi(x) = x^p$, $p > 1$, the bound of the lemma takes an especially simple form, and we obtain the following important corollary.

Corollary 1. Assume that the function $b(z)$ in Eq. (2.11) satisfies Eqs. (2.12) and (2.13). For every $\beta \in [1/2, 1]$ and for every $p \geqslant 1$,

$$\begin{aligned}|u|^{-\zeta} A_\beta[|v|^{2p}] \leqslant\; &-(1-\gamma_p)\,(|v|^{2p} + |w|^{2p}) \\ &+ \gamma_p\left((|v|^2 + |w|^2)^p - |v|^{2p} - |w|^{2p}\right),\end{aligned}$$

where the constant $\gamma_p > 0$ is known explicitly, satisfies $\gamma_1 = 1$, $\lim_{p\to\infty}\gamma_p = 0$, and is strictly decreasing for $p > 1$. Furthermore, if the function $b(\cos\vartheta)$ in Eq. (2.11) is bounded, we obtain the estimate

$$\gamma_p < \min\left\{1, \tfrac{16\pi b^*}{p+1}\right\}, \quad p > 1,$$

where $b^* = \max\limits_{-1\leqslant z\leqslant 1} b(z)$.

Proof. Taking $\Psi(x)=x^p$, $p>1$, and applying the result of the previous lemma we obtain

$$\begin{aligned}|u|^{-\zeta}A_\beta[|v|^{2p}] &\leqslant \gamma_p\,(|v|^2+|w|^2)^p-|v|^{2p}-|w|^{2p}\\ &= -(1-\gamma_p)\,(|v|^{2p}+|w|^{2p})\\ &\quad+\gamma_p\left((|v|^2+|w|^2)^p-|v|^{2p}-|w|^{2p}\right),\end{aligned}$$

where

$$\gamma_p=4\pi\int_{-1}^{1}\left(\frac{1+\mu}{2}\right)^p h_\beta(\mu)\,d\mu, \tag{3.15}$$

and h_β is as in Lemma 1. For $p=1$ we can compute

$$\gamma_1=2\pi\int_{-1}^{1}\left(\left(\frac{1+\mu}{2}\right)+\left(\frac{1-\mu}{2}\right)\right)h_\beta(\mu)\,d\mu=2\pi\int_{-1}^{1}h_\beta(\mu)\,d\mu=1.$$

Also, since $\left(\frac{1+\mu}{2}\right)^p$ is strictly decreasing for $p>1$, pointwise in $\mu\in(-1,1)$, we see that γ_p is strictly decreasing. Further, if $b(\cos\vartheta)$ is bounded, we have

$$h_\beta(\mu)\leqslant\left(1+\left(\tfrac{1}{\beta}-1\right)^2\right)\max_{-1\leqslant z\leqslant 1}b(z)\leqslant 2b^*. \tag{3.16}$$

Using the bound Eq. (3.16) in Eq. (3.15) we obtain the required estimate for γ_p and complete the proof. ∎

Remark 4. For the hard sphere model, the expression (3.15) for the constant γ_p simplifies in the cases $\beta=1$ (elastic interactions), when

$$\gamma_p=\frac{2}{p+1},$$

and $\beta=1/2$ ("sticky" particles):

$$\gamma_p=\frac{p2^p+1}{2^{p-2}(p+1)(p+2)}.$$

For general β the integrand of Eq. (3.15) is too complicated to yield an answer in closed form, and we have to rely on the established inequality for γ_p, which for $b(\cos\vartheta)=\frac{1}{4\pi}$ becomes

$$\gamma_p<\min\left\{1,\ \tfrac{4}{p+1}\right\}.$$

The crucial property of this estimate, which will allow us to obtain the results formulated in Section 2 is the "inverse first power" decay of γ_p for p large.

4. MOMENT INEQUALITIES

Our further goal is to establish a system of moment inequalities in which the moments of higher order are estimated in terms of the lower ones. The estimate of Corollary 1 is the main instrument in obtaining such a result. Its crucial property is that the first term on the right-hand side is negative and of higher order in $|v|$ and $|w|$ than the second one: for integer p this can be verified easily using the binomial formula (cf. ref. 14). To give to this observation a precise sense for non-integer p we establish in next lemma simple estimates involving truncated binomial expansions.

The presentation here and in all the sequel will be restricted to the case of the hard-sphere kernels ($\zeta = 1$ and $b(\cos\theta) = \frac{1}{4\pi}$ in 2.11).

Lemma 2. Assume that $p > 1$, and let k_p denote the integer part of $\frac{p+1}{2}$. Then for all $x,\ y > 0$ the following inequalities hold

$$\begin{aligned} \sum_{k=1}^{k_p-1} \binom{p}{k} (x^k y^{p-k} + x^{p-k} y^k) &\leqslant (x+y)^p - x^p - y^p \\ &\leqslant \sum_{k=1}^{k_p} \binom{p}{k} (x^k y^{p-k} + x^{p-k} y^k). \end{aligned} \tag{4.1}$$

Remarks 1. 1) The binomial coefficients for non-integer p are defined as

$$\binom{p}{k} = \frac{p(p-1)\dots(p-k+1)}{k!}, \quad k \geqslant 1; \qquad \binom{p}{0} = 1.$$

2) In the case when p is an odd integer the first of the inequalities becomes an equality and coincides with the binomial expansion of $(x+y)^p$.

Proof. The proof will be achieved by induction on $n = k_p = 1, 2, 3\dots$ In the case $k_p = 0$ the following inequality is satisfied for $-1 < p \leqslant 1$:

$$(x+y)^p - x^p - y^p \leqslant 0.$$

Next, for $n = 1$ and $1 < p \leqslant 3$, using the above inequality and the identity

$$0 \leqslant (x+y)^p - x^p - y^p = \int_0^x \int_0^y p(p-1)(t+\tau)^{p-2}\, d\tau\, dt,$$

we obtain

$$
\begin{aligned}
(x+y)^p - x^p - y^p &\leqslant \int_0^x \int_0^y p(p-1)(t^{p-2}+\tau^{p-2})\,d\tau\,dt \\
&= p\,(xy^{p-1}+x^{p-1}y),
\end{aligned}
$$

which provides the basis for the induction.

Assuming now that the inequalities (4.1) are true for $2n-1 < p \leqslant 2n+1$, we write

$$
(x+y)^{p+2} - x^{p+2} - y^{p+2} = \int_0^x \int_0^y (p+2)(p+1)(t+\tau)^p\,d\tau\,dt. \tag{4.2}
$$

By the induction hypothesis, the right-hand side of Eq. (4.2) is bounded from below by

$$
\begin{aligned}
&\textstyle\int_0^x \int_0^y (p+2)(p+1)(t^p+\tau^p)\,d\tau\,dt \\
&\quad \textstyle+ \int_0^x \int_0^y (p+2)(p+1) \sum_{k=1}^{k_p-1} \binom{p}{k} (t^k \tau^{p-k} + t^{p-k}\tau^k)\,d\tau\,dt,
\end{aligned}
$$

and from above by a similar expression with k_p-1 replaced by k_p. Performing the integration, using the identity

$$
\frac{(p+2)(p+1)}{(k+1)(p-k+1)} \binom{p}{k} = \binom{p+2}{k+1},
$$

and noticing that $k_p+1 = k_{p+2}$, we obtain the lower bound for Eq. (4.2) in the form

$$
\begin{aligned}
&(p+2)(xy^{p+2}+x^{p+2}y) + \textstyle\sum_{k=1}^{k_p-1} \binom{p+2}{k+1} (x^{k+1}y^{p+1-k} + x^{p+1-k}y^{k+1}) \\
&\quad = \textstyle\sum_{k=1}^{k_{p+2}-1} \binom{p+2}{k} (x^k y^{p+2-k} + x^{p+2-k}y^k),
\end{aligned}
$$

and the upper bound with $k_{p+2}-1$ replaced by k_{p+2}. This completes the induction argument. ∎

In next lemma we obtain estimates of the moments of the collision terms Q_p (2.22) in terms of the moments m_p of the distribution function.

Lemma 3. For every $p > 1$,

$$
-m_{p+1/2} \leqslant Q_p \leqslant -(1-\gamma_p)\,m_{p+\frac{1}{2}} + \gamma_p\,S_p\,,
$$

where

$$S_p = \sum_{k=1}^{k_p} \binom{p}{k} \left(m_{k+\frac{1}{2}} m_{p-k} + m_k m_{p-k+\frac{1}{2}} \right) \tag{4.3}$$

and γ_p is the constant from Corollary 1.

Proof. Multiplying the inequality of Corollary 1 by $f(v)f(w)\,|v-w|$ and integrating with respect to v and w, we obtain

$$\begin{aligned} Q_p \leqslant\ & \frac{\gamma_p}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(v) f(w)\, |v-w| \left((|v|^2 + |w|^2)^p - |v|^{2p} - |w|^{2p} \right) dv\, dw \\ & -(1-\gamma_p) \int_{\mathbb{R}^3} f(v)\, |v|^{2p} \int_{\mathbb{R}^3} f(w)\, |v-w|\, dv\, dw. \end{aligned} \tag{4.4}$$

The inner integral in the last term can be estimated as

$$\int_{\mathbb{R}^3} f(w)\, |v-w|\, dw \geqslant |v|.$$

The last inequality follows by Jensen's inequality, since f is normalized to have unit mass and zero mean, and as the function $|v-w|$ is convex in w for every v fixed. Thus, the last integral term in Eq. (4.4) is estimated from below by

$$\int_{\mathbb{R}^3} f(v)\, |v|^{2p+1}\, dv = m_{p+1/2}.$$

In the first integral term in Eq. (4.4) we use the inequality $|v-w| \leqslant |v| + |w|$ and the upper estimate of Lemma 2 to get

$$\begin{aligned} & |v-w| \left((|v|^2 + |w|^2)^p - |v|^{2p} - |w|^{2p} \right) \\ & \quad \leqslant \sum_{k=1}^{k_p} \binom{p}{k} \left(|v|^{2(k+1/2)} |w|^{2(p-k)} + |v|^{2(p-k+1/2)} |w|^{2k} \right). \end{aligned} \tag{4.5}$$

Substituting the estimate Eq. (4.5) into Eq. (4.4) and performing the integration we obtain the upper bound of the Lemma.

For the lower bound we use Eq. (2.6), neglect the nonnegative $\mathcal{Q}^+$ term and estimate the moments of $\mathcal{Q}^-$ in the same way as we did for the second integral term in Eq. (4.4). This completes the proof. ∎

Assuming suitable smoothness and decay conditions on f we can calculate the moments of the forcing terms as follows:

Pure diffusion (2.2):

$$G_p = \int_{\mathbb{R}^3} f(v)\, \mu\, \Delta |v|^{2p}\, dv = 2\mu\, p\,(2p+1)\; m_{p-1}\,. \tag{4.6}$$

Diffusion with friction (2.3):

$$\begin{aligned} G_p &= \textstyle\int_{\mathbb{R}^3} f(v)\,(\mu\, \Delta |v|^{2p} - \lambda\, v \cdot \nabla |v|^{2p})\, dv \\ &= -2\lambda\, p\, m_p + 2\mu\, p\,(2p+1)\; m_{p-1}\,. \end{aligned} \tag{4.7}$$

Self-similar solutions (2.4):

$$G_p = 2\kappa\, p\, m_p. \tag{4.8}$$

Shear flow term (2.5):

$$G_p = 2\kappa\, p \int_{\mathbb{R}^3} f(v)\, v_1\, v_2\, |v|^{2p-2}\, dv \leqslant 2\kappa\, p\, m_p\,. \tag{4.9}$$

We further use the steady moment equations

$$Q_p + G_p = 0\,, \tag{4.10}$$

obtained from Eq. (2.21), together with the bounds of Lemma 3 to obtain, in the cases Eq. (4.6)–Eq. (4.8) the following *moment inequalities*:

$$G_p \leqslant m_{p+\frac{1}{2}} \leqslant \frac{1}{1-\gamma_p}\big(G_p + \gamma_p\, S_p\big), \quad \text{for every} \quad p > 1, \tag{4.11}$$

and in the case Eq. (4.9),

$$m_{p+\frac{1}{2}} \leqslant \frac{1}{1-\gamma_p}\big(2\kappa p\, m_p + \gamma_p\, S_p\big), \quad \text{for every} \quad p > 1, \tag{4.12}$$

where S_p is given by Eq. (4.3).

At this point we can make an important observation that since the terms G_p and S_p depend on the moments m_k of order at most p ($p - 1/2$ in the case of S_p), inequalities ((4.11) and (4.12)) can be "solved" recursively. More precisely, assuming some properties for the moments of lower order we can use the recursive inequalities to obtain information about the behavior of the moments m_p, for p large. We will therefore use the structure of inequalities (4.11), (4.12) to study the growth of the normalized moments $\big(m_{sk/2}/k!\big)$, which will provide the crucial information about the convergence of the series (2.20).

5. INEQUALITIES FOR NORMALIZED MOMENTS

To achieve the goal described in the end of the previous section, we substitute $z_{sk/2} = m_{sk/2}/k!$ in the moment inequalities and study the resulting system of inequalities for the normalized moments. To simplify the notation and to allow for a bit more flexibility we set

$$z_p = \frac{m_p}{\Gamma(ap+b)}, \quad p \geqslant 0, \tag{5.1}$$

where $a = 2/s$ and b is a constant to be determined. Generally, as we see below, for each of the cases described above there is a particular value of a, such that the family (z_p) with $p \geqslant 0$ has geometric (exponential) growth for p large, for every $b > 0$. It is convenient to leave the value of the parameter b free, since by choosing b in a sensible way we will be able to simplify the inequalities satisfied by z_p.

We shall first look for estimates of the term S_p in the moment inequalities ((4.11) and (4.12)), expressed in terms of the normalized moments z_p. We recall the definition of the beta function,

$$B(x, y) = \int_0^1 s^{x-1}(1-s)^{y-1}\, ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{5.2}$$

which will be used in the proof of next lemma.

Lemma 4. Let $m_p = z_p\, \Gamma(ap+b)$ with $a \geqslant 1$ and $b > 0$. Then for every $p > 1$,

$$S_p \leqslant A\, \Gamma\Big(ap + \frac{a}{2} + 2b\Big) Z_p,$$

where

$$Z_p = \max_{1 \leqslant k \leqslant k_p} \big\{ z_{k+1/2} z_{p-k},\, z_k z_{p-k+1/2} \big\} \tag{5.3}$$

and $A = A(a, b)$ is a constant independent of p.

Proof. Substituting Eq. (5.1) in the expression Eq. (4.3) for S_p we get

$$\begin{aligned} S_p = \sum_{k=1}^{k_p} \binom{p}{k} \Big(& \Gamma\Big(ak + \frac{a}{2} + b\Big)\, \Gamma(a(p-k)+b)\, z_{k+1/2}\, z_{p-k} \\ & + \Gamma(ak+b)\, \Gamma\Big(a(p-k) + \frac{a}{2} + b\Big)\, z_k\, z_{p-k+1/2} \Big). \end{aligned} \tag{5.4}$$

Using Eq. (5.2), we can rewrite Eq. (5.4) as

$$\Gamma(ap+\tfrac{a}{2}+2b)\sum_{k=1}^{k_p}\binom{p}{k}\Big(B\big(ak+\tfrac{a}{2}+b,a(p-k)+b\big)\,z_{k+1/2}\,z_{p-k} \\ +B\big(ak+b,a(p-k)+\tfrac{a}{2}+b\big)\,z_k\,z_{p-k+1/2}\Big). \tag{5.5}$$

Next, we estimate the products $z_{k+1/2}\,z_{p-k}$ and $z_k\,z_{p-k+1/2}$ by their maximum Z_p, obtaining the following bound for the sum in Eq. (5.5)

$$Z_p\sum_{k=1}^{k_p}\binom{p}{k}\Big(B\big(ak+\tfrac{a}{2}+b,a(p-k)+b\big)+B\big(ak+b,a(p-k)+\tfrac{a}{2}+b\big)\Big) \\ =Z_p\int_0^1 s^{\frac{a}{2}+b-1}(1-s)^{b-1}\sum_{k=1}^{k_p}\binom{p}{k}\left\{s^{ak}(1-s)^{a(p-k)}+s^{a(p-k)}(1-s)^{ak}\right\}ds. \tag{5.6}$$

Since the expression in braces depends monotonically on a, we estimate it from above by setting $a=1$. Further, using the lower bound of Lemma 2, the right-hand side of Eq. (5.6) is bounded from above by

$$Z_p\int_0^1\Big\{s^{\frac{a}{2}+b-1}(1-s)^{b-1}\,(1-s^p-(1-s)^p) \\ +\chi_p\binom{p}{k_p}s^{\frac{a}{2}+b-1}(1-s)^{b-1}\big(s^{k_p}(1-s)^{p-k_p}+s^{p-k_p}(1-s)^{k_p}\big)\Big\}\,ds\,, \tag{5.7}$$

where $\chi_p=0$ if p is an odd integer, and 1, otherwise. Neglecting the negative terms in $1-s^p-(1-s)^p$ and using the definition of the beta function again, we obtain the bound

$$B\big(\tfrac{a}{2}+b,b\big) \\ +\chi_p\binom{p}{k_p}\Big(B\big(k_p+\tfrac{a}{2}+b,\,p-k_p+b\big)+B\big(k_p+b,\,p-k_p+\tfrac{a}{2}+b\big)\Big). \tag{5.8}$$

The first term in Eq. (5.8) is a constant independent on p; to estimate the second term we recall the following asymptotic formula for the gamma function:[(1)]

$$\lim_{p\to\infty}\frac{\Gamma(p+r)}{\Gamma(p+s)}\,p^{s-r}=1\,, \tag{5.9}$$

for all r, $s > 0$. Therefore, taking the first beta function in the second term of 5.8 for definiteness, we obtain

$$\binom{p}{k_p} B\big(k_p+\frac{a}{2}+b,\, p-k_p+b\big) = \frac{\Gamma(p+1)\,\Gamma(k_p+\frac{a}{2}+b)\,\Gamma(p-k_p+b)}{\Gamma(p+\frac{a}{2}+2b)\,\Gamma(k_p+1)\,\Gamma(p-k_p+1)}$$
$$\leqslant C\, p^{1-\frac{a}{2}-2b} k_p^{\frac{a}{2}+b-1} (p-k_p)^{b-1}\,.$$

A similar inequality can be obtained for the other beta function term. It is clear now that the second term in Eq. (5.8) is $O(p^{-1})$ for $p \to \infty$, and since it also is locally bounded for $p \geqslant 0$, it is bounded uniformly in p. Denoting now by $A = A(a,b)$ the uniform bound of 5.8 we obtain the conclusion of the lemma. ∎

Remark 5. A more careful analysis of the expression (5.6) would provide a sharper upper bound

$$C\, p^{-a} Z_p \tag{5.10}$$

for that expression, at least for $1 \leqslant a \leqslant 2$. Thus, the factor $\Gamma\big(ap+a/2+b\big)$ in the estimate of the lemma could be improved to $\Gamma\big(ap-a/2+b\big)$. However, the result in the present formulation will be sufficient to obtain the necessary bounds for the moments, so we will not pursue the improved estimates based on the bound (5.10).

We further apply Lemma 4 to simplify the inequalities satisfied by the normalized moments (5.1). Substituting Eq. (5.1) in Eq. (4.11) and using Lemma 4 yields in the case of pure diffusion 4.6

$$\begin{aligned} &2\mu\, \frac{\Gamma(ap-a+b)}{\Gamma(ap+\frac{a}{2}+b)}\, p(2p+1)\, z_{p-1} \\ &\leqslant z_{p+\frac{1}{2}} \\ &\leqslant \frac{2\mu}{1-\gamma_p}\, \frac{\Gamma(ap-a+b)}{\Gamma(ap+\frac{a}{2}+b)}\, p(2p+1)\, z_{p-1} \\ &\quad + \frac{\gamma_p A}{1-\gamma_p}\, \frac{\Gamma(ap+\frac{a}{2}+2b)}{\Gamma(ap+\frac{a}{2}+b)}\, Z_p\,, \end{aligned} \tag{5.11}$$

for all $p \geqslant 1$. In the case of diffusion with friction 4.7, the terms

$$-2\lambda\, \frac{\Gamma(ap+b)}{\Gamma(ap+\frac{a}{2}+b)}\, p\, z_p \quad \text{and} \quad -\frac{2\lambda}{1-\gamma_p}\, \frac{\Gamma(ap+b)}{\Gamma(ap+\frac{a}{2}+b)}\, p\, z_p \tag{5.12}$$

will be added to the left and the right-hand sides of Eq. (5.11), respectively. For the shear flow case (4.9) we obtain

$$z_{p+\frac{1}{2}} \leqslant \frac{2\kappa}{1-\gamma_p} \frac{\Gamma(ap+b)}{\Gamma(ap+\frac{a}{2}+b)} \, p \, z_p + \frac{\gamma_p A}{1-\gamma_p} \frac{\Gamma(ap+\frac{a}{2}+2b)}{\Gamma(ap+\frac{a}{2}+b)} Z_p \,. \tag{5.13}$$

Using Corollary 1, for every $\varepsilon > 0$ and for all $p > 1+\varepsilon$, the constants involving γ_p can be estimated as follows:

$$1 \leqslant \frac{1}{1-\gamma_p} \leqslant \frac{1}{1-\gamma_{1+\varepsilon}} = K_\varepsilon \tag{5.14}$$

and

$$\frac{\gamma_p}{1-\gamma_p} \leqslant \frac{4K_\varepsilon}{p+1}. \tag{5.15}$$

Further, using the identities

$$z\,\Gamma(z) = \Gamma(z+1) \quad \text{and} \quad z\,(z+1)\,\Gamma(z) = \Gamma(z+2)$$

and estimating

$$0 < c_3 \leqslant \frac{2p\,(2p+1)}{(ap-a+b)(ap+1-a+b)} \leqslant C_3,$$

and

$$ap + \frac{a}{2} + 2b - 1 \leqslant C_4 \, \frac{p+1}{4},$$

we can reduce the inequalities (5.11) to

$$\begin{aligned} c_3\,\mu\,\tfrac{\Gamma(ap-a+b+2)}{\Gamma(ap+\frac{a}{2}+b)}\, z_{p-1} \leqslant z_{p+\frac{1}{2}} \leqslant C_3\,K_\varepsilon\,\mu\,\tfrac{\Gamma(ap-a+b+2)}{\Gamma(ap+\frac{a}{2}+b)}\, z_{p-1} \\ + C_4\,K_\varepsilon\,\tfrac{\Gamma(ap+\frac{a}{2}+2b-1)}{\Gamma(ap+\frac{a}{2}+b)}\, Z_p\,. \end{aligned} \tag{5.16}$$

For the additional terms (5.12) appearing in the equation with friction, we use the inequalities

$$c_5 \leqslant \frac{2p}{ap+b} \leqslant C_5$$

to estimate them as

$$-C_5 K_\varepsilon \lambda \frac{\Gamma(ap+b+1)}{\Gamma(ap+\frac{a}{2}+b)}\, z_p \qquad \text{and} \qquad -c_5\lambda \frac{\Gamma(ap+b+1)}{\Gamma(ap+\frac{a}{2}+b)}\, z_p. \tag{5.17}$$

Finally, for the self-similar solution case we obtain the inequalities

$$\begin{aligned} c_5\,\kappa\,\tfrac{\Gamma(ap+b+1)}{\Gamma(ap+\frac{a}{2}+b)}\,z_p \leqslant z_{p+\frac{1}{2}} \;\leqslant\; & C_5\,K_\varepsilon\,\kappa\,\tfrac{\Gamma(ap+b+1)}{\Gamma(ap+\frac{a}{2}+b)}\,z_p \\ & + C_4\,K_\varepsilon\,\tfrac{\Gamma(ap+\frac{a}{2}+2b-1)}{\Gamma(ap+\frac{a}{2}+b)}\,Z_p\,, \end{aligned} \tag{5.18}$$

the last of which is also true in the shear flow case.

We shall next look specifically at the cases of the exponents s appearing in Theorems 1 and 2, and the corresponding values of $a=\frac{2}{s}$. In the case of pure diffusion we take $a=\frac{4}{3}$ and the inequalities (5.16) take the form

$$c_3\,\mu\,z_{p-1} \leqslant z_{p+\frac{1}{2}} \leqslant C_3\,K_\varepsilon\,\mu\,z_{p-1} + C_4\,K_\varepsilon\,\frac{\Gamma(\frac{4}{3}\,p+\frac{2}{3}+2b-1)}{\Gamma(\frac{4}{3}\,p+\frac{2}{3}+b)}\,Z_p\,, \tag{5.19}$$

for every $p>1+\varepsilon$. We notice that if $b<1$, the asymptotic formula (5.9) allows us to control the factor in front of the Z_p term in Eq. (5.19) in the following way:

$$C_4\,K_\varepsilon\,\frac{\Gamma(\frac{4}{3}\,p+\frac{2}{3}+2b-1)}{\Gamma(\frac{4}{3}\,p+\frac{2}{3}+b)} \leqslant \frac{1}{2}, \quad \text{for} \quad p \geqslant p_1, \tag{5.20}$$

if we take p_1 sufficiently large. Inequality (5.19) then becomes

$$c_3\,\mu\,z_{p-1} \leqslant z_{p+\frac{1}{2}} \leqslant C_3\,K_\varepsilon\,\mu\,z_{p-1} + \frac{1}{2}\,Z_p, \quad \text{for} \quad p \geqslant p_1. \tag{5.21}$$

In the case of diffusion with friction the choice $a=1$ gives us the inequalities

$$\begin{aligned} -C_5\,K_\varepsilon\,\lambda\,z_p + c_3\,\mu\,z_{p-1} \leqslant \tfrac{\Gamma(p+\frac{1}{2}+b)}{\Gamma(p+1+b)}\,z_{p+\frac{1}{2}} \leqslant & -c_5\lambda\,z_p \\ & + C_3\,K_\varepsilon\,\mu\,z_{p-1} + C_4\,K_\varepsilon\,\tfrac{\Gamma(p-\frac{1}{2}+2b)}{\Gamma(p+b+1)}\,Z_p\,. \end{aligned} \tag{5.22}$$

Taking now b < 3/2 and choosing p_1 large enough, we obtain using Eq. (5.9),

$$C_4 \frac{\Gamma(p-\frac{1}{2}+2b)}{\Gamma(p+b+1)} \leqslant \frac{c_5\lambda}{2} \quad \text{and} \quad 0 \leqslant \frac{\Gamma(p+\frac{1}{2}+b)}{\Gamma(p+1+b)} \leqslant 1, \quad \text{for} \quad p \geqslant p_1. \tag{5.23}$$

We can then use Eq. (5.22) to obtain the following simple inequalities

$$C_5\, K_\varepsilon\, \lambda\, z_p \geqslant c_3\, \mu\, z_{p-1} - z_{p+\frac{1}{2}} \tag{5.24}$$

and

$$c_5\, \lambda\, z_p \leqslant C_3\, K_\varepsilon\, \mu\, z_{p-1} + \frac{1}{2} c_5\, \lambda\, Z_p\,, \tag{5.25}$$

for all $p \geqslant p_1$.

Finally, in the case of self-similar solutions we take $a = 2$, and Eq. (5.18) becomes

$$c_5\, \kappa\, z_p \leqslant z_{p+\frac{1}{2}} \leqslant C_5\, K_\varepsilon\, \kappa\, z_p + C_4\, K_\varepsilon\, \frac{\Gamma(2p+2b)}{\Gamma(2p+b+1)}\, Z_p\,. \tag{5.26}$$

We then take $b < 1$ and choose p_1 large enough to obtain

$$C_4\, K_\varepsilon\, \frac{\Gamma(2p+2b)}{\Gamma(2p+b+1)} \leqslant \frac{1}{2}, \quad \text{for} \quad p \geqslant p_1. \tag{5.27}$$

Inequality (5.26) then simplifies to

$$c_5\, \kappa\, z_p \leqslant z_{p+\frac{1}{2}} \leqslant C_5\, K_\varepsilon\, \kappa\, z_p + \frac{1}{2} Z_p\,, \quad \text{for} \quad p \geqslant p_1. \tag{5.28}$$

The second of these inequalities is also satisfied in the shear flow case.

Inequalities (5.21), (5.24), (5.25) and (5.28) give a simple and clear picture of the balance between various terms in the moment inequalities for large p. One can easily identify in these inequalities the "loss terms" (moments of order $p+1/2$), the non-cancelled parts of the "gain terms" (terms involving Z_p), diffusion (moments of order $p-1$) and friction-like terms (moments of order p). It is also easy to track down the dependence on the parameter $a = 2/s$) in the inequalities used in the previous step of the derivation. In fact, one could easily reverse our approach and without any *a priori* knowledge about the values of s appearing in Theorems 1 and 2 obtain them just on the base of the inequalities (5.16)–(5.18), as the only values for which the "correct" geometric growth of z_p may be obtained. Of course, this is not necessary for the purpose of the present study; however, such an approach may prove to be useful in other situations when formal arguments do not yield immediate answers.

6. PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. We will establish the following statement that will imply the conclusion of the Theorem (see also the Remark after Theorem 2): for every $p_0 > 1$ there are positive constants c, q, depending on m_0 and m_1 only, and C, Q, depending on m_0, p_0 and m_{p_0} only, such that

$$c\,q^k \leqslant \frac{m_{\frac{sk}{2}}}{k!} \leqslant C\,Q^k, \tag{6.1}$$

for all $k \geqslant \frac{2}{s}$, where $s = \frac{3}{2}$ in the case the pure diffusion, $s = 2$ for diffusion with friction, and $s = 1$ for the self-similar solutions 2.4. Equivalently, with $a = \frac{2}{s}$ and z_p as in Eq. (5.1), we can set $a = \frac{4}{3}$, $a = 1$ and $a = 2$ in the respective cases and look for estimates

$$c\,q^p \leqslant z_p \leqslant C\,Q^p, \tag{6.2}$$

for all $p \geqslant 1$. (The values of constants c and C in Eq. (6.2) are generally different from those in Eq. (6.1).)

Notice that it would be sufficient to prove Eq. (6.2) for a *certain* value of $b > 0$ in the definition of z_p (5.1). Indeed, since

$$C_1\,p^{b_1-b_2} \leqslant \frac{\Gamma(ap+b_1)}{\Gamma(ap+b_2)} \leqslant C_2\,p^{b_1-b_2},$$

changing the value of b in Eq. (5.1) essentially results in the multiplication of z_p by the factor $Cp^{b_1-b_2}$, which can be compensated for by adjusting the constants in Eq. (6.2). We fix the value of $b < 1$ so that inequalities Eqs. (5.19), (5.21) and (5.22) are available for p sufficiently large.

The proof of the inequalities (6.2) is accomplished in two steps. First, we show that Eq. (6.2) holds on the initial interval, $1 \leqslant p \leqslant p_1$, where p_1 (dependent on p_0 and b) is chosen so that inequalities (5.20) and (5.23) hold with $\varepsilon = \frac{1-p_0}{2}$.

Step 1: Initial interval. We notice that for $1 \leqslant p \leqslant p_1$, the gamma function is bounded both from above and from below:

$$0 < c_0 \leqslant \frac{1}{\Gamma(ap+b)} \leqslant C_0, \tag{6.3}$$

where for $a>0$ and $b>0$ the constants c_0 and C_0 depend only on a, b and p_0. Thus, on the initial interval it suffices to estimate m_p instead of z_p in Eq. (6.2).

To obtain the desired estimate, we first use Jensen's inequality to derive for every $0<p'<p<p''$ the inequalities

$$\left(m_{p'}^{1/p'}\right)^p \leqslant m_p \leqslant \left(m_{p''}^{1/p''}\right)^p. \tag{6.4}$$

Taking $p'=1$ and $p''=p_0$ we obtain the bounds

$$c\, q^p \leqslant m_p \leqslant C\, Q^p \tag{6.5}$$

for $1 \leqslant p \leqslant p_0$, with $c=C=1$, $q=m_1$ and $Q=Q_0=\max\{1, m_{p_0}^{1/p_0}\}$.

Step 1: Pure diffusion. We take $\varepsilon = \frac{p_0-1}{2}$, use the bounds Eqs. (5.14) and (5.15) in the moment inequalities (4.11), (4.6) and estimate

$$S_p \leqslant 2^{p+1} M_p\,, \quad \text{where} \quad M_p = \max_{1 \leqslant k \leqslant k_p} \left\{ m_k\, m_{p-k+\frac{1}{2}}, m_{k+\frac{1}{2}}\, m_{p-k} \right\}. \tag{6.6}$$

We then obtain, for all $p > 1+\varepsilon$, the inequalities

$$2\,\mu\,(2p+1)\, m_{p-1} \leqslant m_{p+\frac{1}{2}} \leqslant 2\, K_\varepsilon\, \mu\, p\,(2p+1)\, m_{p-1} + K_\varepsilon 2^{p+1} M_p\,. \tag{6.7}$$

Now we see that using Eq. (6.7) we can extend the bounds Eq. (6.5) (by augmenting the constants q and Q if necessary) to the interval $\frac{3}{2} + \varepsilon \leqslant p \leqslant p_0 + \frac{1}{2}$. Using the interpolation inequality (6.4) we can then extend the bounds Eq. (6.5) to all intermediate values $p_0 < p < p_0 + \frac{1}{2}$.

Further, by iterating inequalities (6.7) we can cover the interval $p_0 \leqslant p \leqslant p_1$ by a fixed number of subintervals of length at most $\frac{1}{2}$, so that finally inequalities (6.5), with the constants depending on m_0, m_1, p_0 and m_{p_0} only, will be extended to the whole interval $1 \leqslant p \leqslant p_1$.

Step 1: Diffusion with friction. We argue as in the previous case and obtain using Eqs. (4.11), and (4.7) the following upper bounds for $m_{p+1/2}$:

$$m_{p+\frac{1}{2}} \leqslant -2\, K_\varepsilon\, \lambda\, p\, m_p + 2\, K_\varepsilon\, \mu\, p\,(2p+1)\, m_{p-1} + K_\varepsilon 2^{p+1} M_p\,, \tag{6.8}$$

for all $p > 1 + \varepsilon$. Neglecting the non-positive friction term on the right-hand side yields the same upper bounds as in the pure diffusion case. On the other hand, the lower bound can be written in the form

$$2\lambda\, p\, m_p + m_{p+\frac{1}{2}} \geqslant 2\mu\, p\,(2p+1)\, m_{p-1}\,, \tag{6.9}$$

which implies that for every $p > 1$ one of the following inequalities is true:

$$2\lambda\, p\, m_p \geqslant \mu\, p\,(2p+1)\, m_{p-1} \quad \text{or} \quad m_{p+\frac{1}{2}} \geqslant \mu\, p\,(2p+1)\, m_{p-1}$$

Combining the two inequalities and using the interpolation inequality (6.4) in the second of the cases we obtain

$$m_p \geqslant \min\left\{ \tfrac{\mu}{\lambda}\,(p+\tfrac{1}{2})\, m_{p-1}\,, \left(\mu\, p\,(2p+1)\, m_{p-1}\right)^{\frac{2p}{2p+1}} \right\}.$$

This allows us to extend the lower bound Eq. (6.5) iteratively to the interval $1 \leqslant p \leqslant p_1$.

Step 1: Self-similar solutions. Using the moment inequalities (4.11), (4.8) and arguing as above we obtain

$$2\kappa\, p\, m_p \leqslant m_{p+\frac{1}{2}} \leqslant 2K_\varepsilon\, \kappa\, p\, m_p + K_\varepsilon\, 2^{p+1}\, M_p\,, \tag{6.10}$$

for all $p \geqslant 1 + \varepsilon$. Using these bounds we extend 6.5 to the interval $1 \leqslant p \leqslant p_1$ by the same iterative argument as in the pure diffusion case.

Step 2: We use the inequalities (5.21), (5.24) and (5.25) to extend bounds Eq. (6.2) to all $p \geqslant 1$ by an induction argument. The base of the induction is established by virtue of the bounds Eqs. (6.5) and (6.3) on the interval $1 \leqslant p \leqslant p_1$. We further verify the induction step separately in each of the three cases.

Step 2: Pure diffusion. Our aim is to find the constants q and Q in such a way that for every $n = 1, 2, 3 \ldots$, the inequalities Eq. (6.2) for $1 \leqslant p \leqslant p_1 + \frac{n-1}{2}$ imply the same inequalities for $p_1 + \frac{n-1}{2} \leqslant p \leqslant p_1 + \frac{n}{2}$. Thus, assuming Eq. (6.2) for $1 \leqslant p \leqslant p_1 + \frac{n-1}{2}$ we use Eq. (5.21) to find

$$c_3\, \mu\, c\, q^{p-1} \leqslant z_{p+\frac{1}{2}} \leqslant C\left(C_3\, K_{p_0}\, \mu\, Q^{-\frac{3}{2}} + \frac{1}{2}\right) Q^{p+\frac{1}{2}}.$$

Taking $q \leqslant (c_3\, \mu)^{\frac{2}{3}}$ and $Q \geqslant (2C_3\, K_{p_0}\, \mu)^{\frac{2}{3}}$ we obtain the inequality

$$c\, q^{p+\frac{1}{2}} \leqslant z_{p+\frac{1}{2}} \leqslant C\, Q^{p+\frac{1}{2}},$$

from which it follows that Eq. (6.2) is true for $p_1 + \frac{n-1}{2} \leqslant p \leqslant p_1 + \frac{n}{2}$.

Step 2: Diffusion with friction. The upper bound case can be treated similarly to the previous one, with the difference that inequalities (5.25) will allow us to increase p by *one* in each step, instead of *one half*, as in the pure diffusion case. Thus, for every $n=1,2,3\ldots$, we assume Eq. (6.2) for $1\leqslant p\leqslant p_1+n-1$ and obtain using Eq. (5.25),

$$z_{p+1}\leqslant\Big(\frac{C_3\,\mu}{C_5\,\lambda}\,Q^{-1}+\frac{1}{2}\Big)\,C\,Q^{p+1}\,.$$

We now take $Q\geqslant\frac{2C_3\mu}{C_5\lambda}$ to obtain the inequalities

$$z_{p+1}\leqslant C\,Q^{p+1},$$

which imply the upper bound Eq. (6.2) for $p_1+n-1\leqslant p\leqslant p_1+n$.

For the lower bound we see that assuming Eq. (6.2) to be true for $1\leqslant p\leqslant p_1+\frac{n-1}{2}$, the inequalities Eq. (5.24) imply that at least one of the inequalities

$$K_\varepsilon\,C_5\,\lambda\,z_p\geqslant\frac{1}{2}\,c_3\,\mu\,c\,q^{p-1}\quad\text{or}\quad z_{p+\frac{1}{2}}\geqslant\frac{1}{2}\,c_3\,\mu\,c\,q^{p-1}$$

is true. By choosing $q<\min\left\{\left(\frac{1}{2}\,c_3\,\mu\right)^{\frac{2}{3}},\,\frac{c_3\,\mu}{2K_\varepsilon\,C_5\,\lambda}\right\}$ we obtain 6.2 for $p_1+\frac{n-1}{2}\leqslant p\leqslant p_1+\frac{n}{2}$.

Step 2: Self-similar solutions. We use inequalities (5.28) and argue as in the pure diffusion case, assuming for every $n=1,2,3\ldots$ that (6.2) holds for $1\leqslant p\leqslant p_1+\frac{n-1}{2}$. We then find

$$c_5\,\kappa\,c\,q^p\leqslant z_{p+\frac{1}{2}}\leqslant\Big(C_5K_\varepsilon\kappa\,Q^{-\frac{1}{2}}+\frac{1}{2}\Big)C\,Q^{p+\frac{1}{2}}.$$

Therefore, taking $q<(c_5\,\kappa)^2$ and $Q>(2C_5K_\varepsilon\kappa)^2$ we obtain Eq. (6.2) for $p_1+\frac{n-1}{2}\leqslant p\leqslant p_1+\frac{n}{2}$.

We now complete the proof of Theorem 1 by an induction argument. ∎

The above proof contains the proof of Theorem 2 as a special case. Indeed the inequalities for the normalized moments in the shear flow case coincide with the upper inequalities for the case of the self-similar solutions. The result of Theorem 2 is weaker than in the latter case, since we were not able to obtain suitable lower bounds for the moments in the shear flow problem.

CONCLUDING REMARKS

The established estimates for the exponential moments give important information about the tails of distribution functions, and further applications of the techniques developed here certainly go beyond the framework of this paper. In particular, an extension of the results to the time-dependent solutions, in the spirit of the paper,[4] is certainly possible. Another promising direction of study stems from the use of the integral bounds together with maximum principles for the Boltzmann equation, in the form suggested by C. Villani.[34] There are indications that such methods may yield more precise forms of asymptotics (in particular, pointwise upper bounds) for some of the problems studied here.[18]

In view of the results of Section 3 it is interesting to notice that an extension of the moment inequalities to more general forms of angular dependence in the collision kernel Eq. (2.11) is very straightforward and does not affect the tail behavior. On the other hand, choosing different values of ζ in Eq. (2.11) one could obtain a variety of different "stretched exponential" tails, with values of s depending on ζ. Again, for every $\zeta > 0$, the chain of the moment inequalities implies that, formally, the tails of any solution with finite moment of an order higher than the kinetic energy are given by "stretched exponential" functions with certain $s > 0$.

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